PDG I
(Zentralübung)

Problem Sheet 7

Question 1

Let $v \in C^2(\mathbb{R})$ (so $N = 1$) and, for $t > 0$, $x \in \mathbb{R}$, define

$$u(t, x) := v\Big(\frac{x}{\sqrt{t}}\Big).$$

(a) Show that

$$u_t = u_{xx}$$

if and only if

$$v''(z) + \frac{z}{2}v'(z) = 0, \quad z \in \mathbb{R}. \tag{1}$$

Show that the general solution of (1) is

$$v(z) = c\int_0^z e^{-s^2/4}\,\mathrm{d}s + d.$$

(b) Differentiate $u(t, x) = v\big(\frac{x}{\sqrt{t}}\big)$ with respect to x and select the constant c properly, to obtain the fundamental solution Φ for the Heat Equation for $N = 1$. Explain why this procedure produces the fundamental solution. (*Hint:* What is the initial condition for u?)

Question 2

(a) Solve the intial value problem for the Heat Equation with convection:

$$\begin{cases} u_t - \Delta u + b \cdot \nabla u = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = g(x) & \text{on } \{t=0\} \times \mathbb{R}^N, \end{cases}$$

where $g\colon \mathbb{R}^N \to \mathbb{R}$ is smooth, and $b \in \mathbb{R}^N$ is a constant. Write your solution in terms of an integral involving the heat kernel $\Phi(\cdot,\cdot)$.

(b) Similarly, write down an explicit solution for the initial value problem

$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = g(x) & \text{on } \{t=0\} \times \mathbb{R}^N, \end{cases}$$

where $g\colon \mathbb{R}^N \to \mathbb{R}$ is smooth, and $c \in \mathbb{R}$ is a constant.

(*Hint for (a) and (b):* Recall methods used previously for the Transport Equation)

(c) Now solve

$$\begin{cases} u_t - \Delta u + b \cdot \nabla u + cu = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = g(x) & \text{on } \{t=0\} \times \mathbb{R}^N, \end{cases}$$

where $g\colon \mathbb{R}^N \to \mathbb{R}$ is smooth, and $b \in \mathbb{R}^N$, $c \in \mathbb{R}$ are constants.

(Note that in the case $n = 1$, this equation is closely linked to the *Black-Scholes Model* for pricing stock options.)

Deadline for handing in: 0800 Wednesday 3 December

Please put solutions in Box 17, 1st floor (near the library)

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`

Sheet 7

① Let $v \in C^2(\mathbb{R})$ and define

$$u(t,x) := v\left(\frac{x}{\sqrt{t}}\right) \qquad t>0, \quad x \in \mathbb{R}.$$

Then
$$u_t(t,x) = -\tfrac{1}{2} x t^{-3/2} v'(t^{-\frac{1}{2}}x)$$
$$u_x(t,x) = t^{-\frac{1}{2}} v'(t^{-\frac{1}{2}}x)$$
$$u_{xx}(t,x) = t^{-1} v''(t^{-\frac{1}{2}}x)$$

(a) If $u_t(t,x) = u_{xx}(t,x)$ (u solves heat eqn in 1d)

Then $-\frac{1}{2} x t^{-3/2} v'(t^{-\frac{1}{2}}x) = t^{-1} v''(t^{-\frac{1}{2}}x)$

Multiply both sides by t: $-\frac{1}{2} x t^{-\frac{1}{2}} v'(t^{-\frac{1}{2}}x) = v''(t^{-\frac{1}{2}}x)$

Write $z = t^{-\frac{1}{2}}x$:

(1) $\quad v''(z) + \frac{z}{2} v'(z) = 0 \qquad z \in \mathbb{R}.$

~~Conversely~~ Also, if (1) holds, then $u_t = u_{xx}$

Solve 1: Write $w(z) = v'(z)$. Then (1) becomes

$w'(z) + \frac{z}{2} w(z) = 0 \qquad 1^{st}$ order ODE.

$$\frac{w'(z)}{w(z)} = -\frac{z}{2}$$

So $\ln(w(z)) = \int -\frac{z}{2}\, dz = -\frac{z^2}{4} + C$

So $w(z) = c\, e^{-z^2/4} = v'(z).$

So $v(z) = c\int_0^z e^{-s^2/4}\, ds + d$

(b) From (a): if u is of the form $u(t,x) = v(t^{-\frac{1}{2}}x)$ and

$$v(z) = c\int_0^z e^{-s^2/4}\,ds + d, \quad \text{- smooth}$$

u solves $u_t = u_{xx}$. $\forall t>0,\ x\in\mathbb{R}$

Also, if (since v is C^3), $(u_x)_t = (u_x)_{xx}$. u_x solves heat eq in 1d.

We want a solution that satisfies $\begin{cases} w_t = w_{xx} \\ w(0,x) = g(x) \end{cases}$ $g \in C(\mathbb{R})$ initial data.

Note that $u_x(t,x) = t^{-\frac{1}{2}} v'(t^{-\frac{1}{2}}x)$

$$= \frac{c}{\sqrt{t}} e^{\frac{-x^2}{4t}}$$

Then $\int_{\mathbb{R}} u_x(t,x-y)\, g(y)\, dy = \frac{c}{\sqrt{t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} g(y)\,dy \longrightarrow \tilde{c}\, g(x)$ at $t\to 0$.

Hence, if we can choose c s.t. $\int_{\mathbb{R}} u_x(t,x)\,dx = 1$,
u_x is fundamental soln of heat equation in 1d.

We want $1 = \frac{c}{\sqrt{t}}\int e^{\frac{-x^2}{4t}}\,dx$

$r = \frac{x}{\sqrt{4t}}$ $\quad = c\,2\int_{\mathbb{R}} e^{-r^2}\,dr = c\sqrt{4\pi}$.

So $c = \frac{1}{\sqrt{4\pi}}$ gives

$$\Phi(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

② For the initial value problem find $u:(0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ satisfy

$$(H)\quad \begin{cases} u_t(t,x) - \Delta_x u(t,x) = 0 & (t,x)\in(0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x) & \text{on } \{0\}\times\mathbb{R}^n \end{cases}$$

When $g:\mathbb{R}^n\to\mathbb{R}$ is smooth, we have a solution

$$u(t,x) = \int_{\mathbb{R}^n} \Phi(t,x-y)\,g(y)\,dy$$

where $\Phi(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-x^2}{4t}}$ is Fundamental Solution for Heat equation

(a) Find a solution to

$$(A)\quad \begin{cases} u_t(t,x) - \Delta_x u(t,x) + b\cdot\nabla_x u(t,x) = 0 & \text{on } (0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x) & \text{on } \{0\}\times\mathbb{R}^n \end{cases}$$

$b\in\mathbb{R}^n$ constant g smooth.

Write $v(t,x) = u(t,x+tb)$

Then $v_t(t,x) = u_t(t,x+tb) + b\cdot\nabla_x u(t,x+tb)$

$\Delta_x v(t,x) = \Delta_x u(t,x+tb)$, $v(0,x) = u(0,x)$

Hence u solves (A) iff v solves (H).

Solution $v(t,x) = \int_{\mathbb{R}^n} \Phi(t,x-y)\,g(y)\,dy = u(t,x+tb)$

So $u(t,x) = \int_{\mathbb{R}^n} \Phi(t,x-y-tb)\,g(y)\,dy$

(b) Find a solution to

$$(B) \begin{cases} u_t(t,x) - \Delta_x u(t,x) + c\,u(t,x) = 0 & \text{in } (0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x) & \text{on } \{0\}\times\mathbb{R}^n \end{cases}$$

$c \in \mathbb{R}$ constant

Write $v(t,x) = e^{ct} u(t,x)$

Then $v_t(t,x) = e^{ct} u_t(t,x) + c e^{ct} u(t,x)$

$\Delta_x v = \Delta_x u$, $v(0,x) = u(0,x)$ $(e^{0t} = 1)$

So u solves (B) $\iff$ $\cancel{e^{ct}}$ v solves (H) ~~with init condits~~ $\cancel{e^{ct}}\, g$

take $v(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y)\, dy = e^{ct} u(t,x)$

So $u(t,x) = e^{-ct} \int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y)\, dy$

(c) Find a solution to

$$(C) \begin{cases} u_t(t,x) - \Delta_x u(t,x) + b\cdot\nabla_x u(t,x) + c\,u(t,x) = 0 \text{ in } (0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x) \text{ on } \{0\}\times\mathbb{R}^n \end{cases}$$

$c \in \mathbb{R}$, $b \in \mathbb{R}^n$ constants

Combine methods in (a) and (b):

Consider $v(t,x) = e^{ct} u(t, x + tb)$

Then u solves (C) $\iff$ v solves (H)

$v(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y)\, dy = e^{ct} u(t, x+tb)$

So $u(t,x) = e^{-ct} \int_{\mathbb{R}^n} \Phi(t, x-y-tb)\, g(y)\, dy$ solves (C).

Black Scholes:

t - time

x - stock price

u - option price